We consider numerical semigroups \( \mathbb{N} \cap I \), for intervals \( I \). We compute the Frobenius number and multiplicity of such semigroups, and show that we may freely restrict \( I \) to be open, closed, or half-open, as we prefer.

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Consider an interval \( I \subseteq \mathbb{Q}^+ \). Set \( S(I) = \{ m \in \mathbb{N} : \exists n \in \mathbb{N}, \frac{m}{n} \in I \} \). This turns out to be a numerical semigroup, and has been the subject of considerable recent investigation (see Chapter 4 of [2] for an introduction). Special cases include modular numerical semigroups (see [4]) where \( I = [\frac{m}{n}, \frac{m}{n} - 1] \) \((m, n \in \mathbb{N})\), proportionally modular numerical semigroups (see [3]) where \( I = [\frac{m}{n}, \frac{m}{n} - s] \) \((m, n, s \in \mathbb{N})\), and opened modular numerical semigroups (see [5]) where \( I = (\frac{m}{n}, \frac{m}{n} - 1) \) \((m, n \in \mathbb{N})\).

We consider instead arbitrary open intervals \( I = (a, b) \). We show that this set of semigroups coincides with the set of semigroups generated by closed and half-open intervals. Consequently, this class of semigroups contains modular numerical semigroups, proportionally modular numerical semigroups, as well as opened modular numerical semigroups. We also compute two important invariants of these numerical semigroups: the Frobenius number \( g(S(I)) \) and multiplicity \( m(S(I)) \).

1. PRELIMINARIES

We begin by defining a helpful function \( \kappa(a, b) \). For \( a, b \in \mathbb{R} \) with \( a < b \) we define \( \kappa(a, b) = \lfloor \frac{b}{b-a} \rfloor \). The function \( \kappa \) has various nice properties, for example \( \kappa(a, b) = \kappa(ac, bc) \) for \( c > 0 \). In the special case of \( a = \frac{m}{n}, b = \frac{m}{n-s} \),
we have \( \kappa(a, b) = \lfloor \frac{n}{s} \rfloor \). The following properties of \( \kappa(a, b) \) are needed in the sequel.

**Lemma 1.1.** Let \( b \neq 0 \). Set \( \kappa = \kappa(a, b) \). If \( \kappa \neq 0 \), then \( \frac{\kappa - 1}{\kappa} \leq \frac{a}{b} \). If \( \kappa \neq -1 \), then \( \frac{a}{b} < \frac{\kappa}{\kappa + 1} \).

**Proof.** We have \( \frac{1}{\kappa} \leq \frac{b}{b - a} < \frac{1}{\kappa + 1} \), and the results follow. If \( \kappa = 0 \), then \( 0 < \frac{a}{b - a} < 1 \) and hence \( \frac{a}{b} < 0 = \frac{\kappa}{\kappa + 1} \). If \( \kappa = -1 \), then \( -1 \leq \frac{b}{b - a} < 0 \) and \( \frac{a}{b} \leq 2 \) so \( \frac{a}{b} \leq 2 = \frac{-1}{\kappa} \).

**Lemma 1.2.** Let \( b > 0 \). Then \( \mathbb{N} \setminus S((a, b)) = \mathbb{N} \cap \bigcup_{n=1}^{\kappa(a, b)} [b(n - 1), an] \).

**Proof.** Because \( S((a, b)) = \mathbb{N} \cap \bigcup_{n=1}^{\infty} (an, bn) \), we have \( \mathbb{N} \setminus S((a, b)) = \mathbb{N} \cap \bigcup_{n=1}^{\infty} [b(n - 1), an] \). Since \( b > 0 \), \( \kappa(a, b) \neq -1 \) and hence by Lemma 1.1, \( b\kappa(a, b) > a(\kappa(a, b) + 1) \). Hence for \( n > \kappa(a, b) \), the intervals are empty and may be excluded.

Lemma 1.2 yields an upper bound for \( g \). This bound will later be improved in Theorem 3.1, but for now this weaker bound suffices.

**Corollary 1.1.** Suppose \( 0 < a < b \). Then \( g(S((a, b))) \leq \lfloor ak(a, b) \rfloor \).

## 2. INTERVALS

We now prove that restricting \( I \) to be open is harmless, as this class of semigroups coincides with ones generated by closed or half-open intervals. To reduce the number of cases to consider, we introduce the symbols \{ , \} to denote endpoints of an interval that are either open or closed. For example, \((a, b)\) indicates an interval that is open on the left. These symbols bind, resolving the ambiguity, when first used; that is, if \((a, b)\) is open then \([a', b)\) means \([a', b)\), and if \((a, b)\) is half-open then \([a', b]\) means \([a', b]\).

The following lemma is the cornerstone of the interval equivalence results.

**Lemma 2.1.** Let \( a \in \mathbb{Q}^{>0} \), \( n \in \mathbb{N} \). Then all rationals in the interval \( \left(a - \frac{a}{nd(a) + 1}, a + \frac{a}{nd(a) + 1}\right) \), other than possibly \( a \), have numerator greater than \( n \).
**Proof.** Suppose \( a = \frac{p}{q} \), so \( d(a) = q \). Consider any rational \( \frac{x}{y} \) with \( 0 < \left| \frac{x}{y} - a \right| < \frac{1}{ny} \), because \( \frac{x}{y} - \frac{p}{q} = \frac{xq - yp}{yq} \geq \frac{1}{yq} \), because \( xq - yp \neq 0 \) since \( \frac{x}{y} \neq a \). Combining, we get \( \frac{p}{q(nq+1)} > \frac{1}{yq} \), hence \( \frac{px}{nq+1} > \frac{x}{y} > a - \frac{a}{nq+1} = \frac{pnq}{nq+1} = \frac{pn}{nq+1} \), and thus \( x > n \). 

**Lemma 2.2.** Suppose that \( I, J, I \cup J \) are all intervals. Then \( S(I \cup J) = S(I) \cup S(J) \). Also, if \( I \subseteq J \), then \( S(I) \subseteq S(J) \) and \( g(S(I)) \geq g(S(J)) \).

**Proof.** Integer \( m \in S(I \cup J) \) if and only if \( \frac{m}{n} \in I \cup J \) for some \( n \). This is true if and only if \( \frac{m}{n} \in I \) or \( \frac{m}{n} \in J \). Hence \( m \in S(I \cup J) \) if and only if \( m \in S(I) \cup S(J) \). If \( I \subseteq J \), then \( S(J) = S(I \cup J) = S(I) \cup S(J) \supseteq S(I) \).

The following theorem allows us to replace a closed endpoint with an open one nearby, leaving the semigroup unchanged. Given a modular or proportionally modular numerical semigroup \( S \), it explicitly gives an open interval \( I \) with \( S(I) = S \).

**Theorem 2.1.** Let \( 0 < a < b \). Then \( S([a, b]) = S([a', b]) \), and \( S([a, b]) = S([a, b']) \), for:

\[
a' = \begin{cases} 
\frac{a - \frac{a}{a(a,b)+1}}{a} & a \in \mathbb{Q} \\
\frac{b + \frac{b}{a(a,b)+1}}{b} & b \in \mathbb{Q} \\
\end{cases}
\]

**Proof.** We consider only \( [a, b] \): \( \{a, b\} \) is symmetric. Suppose first that \( a \notin \mathbb{Q} \). By Lemma 2.2, \( S([a', b]) = S([a', b']) \cup S([a', a']) = S([a', b]) \) since \( S([a', a']) = \emptyset \). We now assume \( a \in \mathbb{Q} \). By Lemma 2.2, \( S([a', b]) = S([a', 2a-2a']) \cup S([a, b]) \). We will show \( S([a', 2a-2a']) \subseteq S([a, b]) \), implying \( S([a', b]) \subseteq S([a, b]) \) and \( S([a', b]) \supseteq S([a, b]) \) by Lemma 2.2.

Let \( c \in S([a', 2a-2a']) \). Hence there is some \( d \in \mathbb{N} \) so that \( \frac{c}{d} \in (a', 2a - a') \). By Lemma 2.1, either \( \frac{c}{d} = a \) (in which case \( c \in S([a, b]) \)), or \( c \notin \{a, b\} \). In the latter case, we apply Corollary 1.1 and \( c > |ak(a, b)| \geq g(S([a, b])) \), so \( c \in S([a, b]) \). 

Theorem 2.2 is a counterpoint to Theorem 2.1, allowing us to replace an open endpoint with a closed one nearby. Proposition 5 in [5] tells us more: that every \( S(I) \) is proportionally modular; that is, there are \( m, n, s \in \mathbb{N} \) where \( S(I) = S([\frac{m}{n}, \frac{m}{n+s}]) \). Unfortunately neither of these results give an explicit formula such as in Theorem 2.1.

**Theorem 2.2.** Let \( 0 < a < b \). Then there are \( a', b' \) with \( S([a, b]) = S([a', b']) \), and \( S([a, b]) = S([a', b]) \). Further, \( \frac{a}{a'}, \frac{b}{b'} \in \mathbb{Q} \).
Proof. We consider only $(a, b)$; $(a, b)$ is symmetric. Suppose first that $a \notin \mathbb{Q}$. By Lemma 2.2, $S([a', b]) = S([a', b]) \cup S([a', a']) = S([a', b])$ since $S([a', a']) = \emptyset$. We now assume $a \in \mathbb{Q}$. Let $a_0$ be any rational in $(a, b)$, and consider the sequence given by $a_i = \frac{a + a_i}{2}$ ($i \geq 1$). By Lemma 2.2, $S([a_1, b]) \subseteq S([a_2, b]) \subseteq \cdots \subseteq S([a, b])$. Set $X = S((a, b)) \setminus S([a_1, b])$. Each $x_j \in X$ has a corresponding $y_j$ with $\frac{x_j}{y_j} \in (a, b)$; since $a_i \rightarrow a$, we have $\frac{x_j}{y_j} \in [a_j, b]$ for some $j$, and hence $x_j \in S([a_j, b])$. Since $|X| < \infty$, we may take $a'$ to be the smallest of these $a_j$. Note that $a' \in \mathbb{Q}$ by construction.

3. CALCULATING $g(S((a, b)))$ AND $m(S((a, b)))$

We now improve on Corollary 1.1 by calculating $g(S((a, b)))$ exactly. Various other results are known for related contexts. For example, if $S([a, b])$ is not a half-line, in [5] it was shown that $g(S((a, b))) = g(S([a, b]))$. Also, if $2 \leq a < b$ with $a, b \in \mathbb{N}$, in [6] it was shown that $g(S((a, b))) = b$.

THEOREM 3.1. Suppose $0 < a < b$. Set $\kappa = \kappa(a, b), \kappa' = \max(\kappa(a - 1, b - 1), 0)$. Then $g(S((a, b))) = \lfloor a\alpha \rfloor$, where $\alpha \in \mathbb{Z}$ satisfies $\kappa' \leq \alpha \leq \kappa$. Specifically,

$$\alpha = \kappa - \sum_{i=\kappa' + 1}^{\kappa} \prod_{j=i}^{\kappa} (1 + |aj| + |bj - j|).$$

Proof. By Lemma 1.2, $g(S((a, b))) = \lfloor a\alpha \rfloor$, for the greatest integer $\alpha$ where $\mathbb{N} \cap [b(\alpha - 1), a\alpha]$ is nonempty; in particular $\alpha \leq \kappa$. The lower bound $\alpha \geq \kappa'$ is trivial when $\kappa' = 0$; if $b \leq 1$ then $\kappa(a - 1, b - 1) = \lfloor \frac{b - 1}{b} \rfloor \leq 0$ and hence $\kappa' = 0$. Otherwise, $b > 1$ and so by Lemma 1.1, $\frac{b - 1}{\kappa'} \leq \frac{b - 1}{\kappa' - 1}$; rearranging we get $b(\kappa'-1) \leq a\kappa' - 1$. Hence the interval $[b(\kappa'-1), a\kappa']$ has length at least 1. It must therefore contain an integer, so $\alpha \geq \kappa'$.

To prove the $\alpha$ formula, for $i \leq \kappa$ we define function $f(i) = \begin{cases} 1 & \alpha < i \\
0 & \alpha \geq i \end{cases}$; this gives us $\alpha = \kappa - \sum_{i=0}^{\kappa} f(i) = \kappa - \sum_{i=\kappa' + 1}^{\kappa} f(i)$. We define $f$ via $f(i) = \prod_{j=i}^{\kappa} \chi(j)$, for $\chi(j) = \begin{cases} 1 & [b(j - 1), aj] \cap \mathbb{N} \neq \emptyset \\
0 & [b(j - 1), aj] \cap \mathbb{N} = \emptyset \end{cases}$. We now have $\alpha = \kappa - \sum_{i=\kappa' + 1}^{\kappa} \prod_{j=i}^{\kappa} \chi(j)$.

We now calculate $\chi(j)$ explicitly by showing that for $j \geq \kappa' + 1$, the interval $[b(j - 1), aj]$ contains at most one integer. For $b \leq 1$, we have $bj > aj \geq aj + (b - 1)$ so $b(j - 1) > aj - 1$. For $b > 1$, by Lemma 1.1 we have $\frac{b - 1}{b} < \frac{\kappa'}{\kappa' + 1} \leq \frac{j - 1}{(j-1)+1}$ for any $j - 1 \geq \kappa'$. Rearranging, we get
b(j - 1) > aj - 1. Hence |[b(j - 1), aj]| \cap \mathbb{N} | \leq 1 and in fact \chi(j) equals the number of integers in [b(j - 1), aj], i.e. \chi(j) = 1 + [aj] + [b(1 - j)].

We have \alpha \in [\kappa', \kappa]; in general, neither bound can be improved. The size of this interval, \kappa - \kappa', can be arbitrarily large, when \( \frac{b}{a} \) is small. On the other hand, the following shows that \kappa - \kappa' is small if \( \frac{b}{a} > 2 \). This is desirable, as it shortens the calculation for \( g \).

**Proposition 3.1.** Let \( 0 < 2a < b \). Let \( \kappa, \kappa' \) be as in Theorem 3.1. Then
\[
\kappa - \kappa' = \begin{cases} 
1 & a < 1 \\
0 & a \geq 1.
\end{cases}
\]

**Proof.** For convenience, set \( I = (\frac{b-1}{b-a}, \frac{b}{b-a}) \); \( \kappa - \kappa' \) counts the number of integers in \( I \). Suppose first that \( b \leq 1 \). Then \( \kappa(a - 1, b - 1) \leq 0 \), so \( \kappa' = 0 \).

Note that \( b > 2a \) implies \( b - a > a \), and hence \( \frac{1}{b-a} < \frac{1}{2} \), so \( 1 + \frac{a}{b-a} < 1 + \frac{a}{2} = 2 \), and hence \( \kappa = [1 + \frac{a}{b-a}] = 1 \). Suppose now that \( a < 1 < b \). If \( a \leq \frac{1}{2} \), then \( b > 1 = \frac{1}{2} + a \). Alternatively, if \( a > \frac{1}{2} \), then \( b > 2a > \frac{1}{2} + a \). Hence \( b > \frac{1}{2} + a \); rearranging we get \( \frac{1}{b-a} < 2 \). Hence \( I \) is of length less than 2, and can contain at most one integer. Therefore \( \kappa - \kappa' \leq 1 \). But \( I \) contains the integer \( 1 = \frac{b-a}{b-a} \), so \( \kappa - \kappa' = 1 \). Lastly, we consider the case \( a \geq 1 \). We have \( b - 1 \geq b - a \), hence \( \frac{b-1}{b-a} \geq 1 \) and \( I \) does not contain 0 or 1. Suppose \( I \) contains integer \( n \geq 2 \). Then \( 2 \leq \frac{b}{b-a} \); rearranging we get \( b \leq 2a \), a contradiction. Hence \( I \) contains no integers, and \( \kappa - \kappa' = 0 \).

Computing \( m(S((a, b))) \) is similar to computing \( g(S((a, b))) \), in that we must count integers in an interval, only this time the intervals are open. We first prove a technical lemma, for which we recall Farey sequences (for an introduction see [1]). The \( n \)th Farey sequence \( F_n \) consists of all reduced fractions in \( [0, 1] \) whose denominator is at most \( n \), arranged in increasing order. The key property we require is that if \( \frac{a}{b}, \frac{c}{d} \) are consecutive terms in a Farey sequence, then \( bc - ad = 1 \).

**Lemma 3.1.** Let \( 0 < a < b \). Let \( n \in \mathbb{N} \) be minimal such that \( (an, bn) \) contains an integer. Suppose \( n > 1 \). Then \( (an, bn) \) contains exactly one integer.

**Proof.** Suppose by way of contradiction that \( (an, bn) \) contains at least two integers. Then there is some \( m \in \mathbb{N} \) such that \( m, m+1 \in (an, bn) \).

Set \( d = \gcd(m, n) \). If \( d > 1 \) then \( m/d \in (an/d, bn/d) \) violates the minimality of \( n \). Similarly, \( \gcd(m + 1, n) = 1 \). Let \( m' \in (0, n - 1) \) with \( m = m' + kn \) for some integer \( k \). We now consider the \( n \)th Farey sequence \( F_n \). Both \( \frac{m'}{n} \) and \( \frac{m'+1}{n} \) are elements of \( F_n \); however \( (m'+1)n - m'n = n > 1 \), so they are not consecutive terms and there must be
We now compute the multiplicity \( m(S((a, b))) \). The reverse problem of finding an open interval whose semigroup possesses a given multiplicity, is solved in [6]. A non-discrete version is proved as Proposition 5 in [3].

**Theorem 3.2.** Suppose \( 0 < a < b \). Set \( \kappa'' = \kappa(1, b - a + 1) \). Then \( m(S((a, b))) = [a\alpha], \) where \( \alpha \in \mathbb{N} \) satisfies \( 1 \leq \alpha \leq \kappa'' \). Specifically,

\[
\alpha = \sum_{i=0}^{\kappa''} \prod_{j=1}^{i} (2 + [aj] + [-bj]).
\]

**Proof.** Let \( \alpha \in \mathbb{N} \) be such that \( m(S((a, b))) \in (a, b); \) then \( m(S((a, b))) = [a\alpha] \). In fact there may be more than one such \( \alpha \) (all yielding the same \( m \)) but we will compute the smallest. By Lemma 1.1, \( \frac{1}{b-a+1} < \frac{\kappa''}{\kappa'' + 1} \). Rearranging, we find \( \kappa''b - \kappa''a > 1 \), so in fact \( \alpha \leq \kappa'' \).

We now prove the \( \alpha \) formula. We proceed in a manner similar to Theorem 3.1, by defining \( f(i) = \begin{cases} 1 & i \leq \alpha \\ 0 & i > \alpha \end{cases} \), via \( f(i) = \prod_{j=1}^{i} (1 - \chi(j)) \), where \( \chi(j) \) is the number of integers in \((aj, bj)\). For \( i < \alpha \), \( \chi(i) = 0 \). By Lemma 3.1, \( \chi(\alpha) = 1 \), so \( f(i) = 0 \) for \( i \geq \alpha \). Hence \( \alpha = \sum_{i=0}^{\kappa''} f(i) = \sum_{i=0}^{\kappa''} \prod_{j=1}^{i} (1 - \chi(j)) \), but \( 1 - \chi(j) = 2 + [aj] + [-bj] \).

We have \( \alpha \in [1, \kappa''] \); in general, neither bound can be improved. The upper bound \( \kappa'' \) can be arbitrarily large, when \( b-a \) is small. On the other hand, the following shows that \( \kappa'' \) is small if \( b-a \) is large, thus simplifying computation of \( m \).

**Proposition 3.2.** Let \( 0 < a < b \). Let \( n \in \mathbb{N} \) be minimal with \( b - a > \frac{1}{n} \). Then \( \kappa'' = n \), in the notation of Theorem 3.2.

**Proof.** We have \( \frac{1}{n} < b - a \leq \frac{1}{n-1} \), hence \( n > \frac{1}{b-a} \geq n-1 \), so \( \lfloor \frac{1}{b-a} \rfloor = n-1 \), and \( \kappa'' = \lfloor \frac{b-a+1}{b-a} \rfloor = \lfloor 1 + \frac{1}{b-a} \rfloor = 1 + (n-1) = n \).

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**References**


